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The Adaption Problem for Nonsymmetric Convex Sets

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We study the problem of optimal recovery in the case of a nonsymmetric convex class of functions. We compare adaptive and nonadaptive methods and prove a bound on how much better adaptive methods can be. We use new inequalities between Gelfand widths and Bernstein widths and new relations between these widths and optimal error bounds for adaptive and nonadaptive methods, respectively.

1. INTRODUCTION

The adaption problem of the title is the following: Let X be a normed space over $\mathbf{R}$ and let $F \subset X$ be a given subset. Assume that $S: X \to Y$ is a continuous linear mapping into another normed space Y that is to be approximated by a method of the form $\varphi \circ N$, where

$$N(f) = (L_1(f), L_2(f), ..., L_n(f)).$$

We assume that the L_k are arbitrary continuous linear functionals $L_k: X \to \mathbf{R}$. We want to minimize the maximal error

$$\Delta_{\max}(\varphi \circ N) = \sup_{f \in F} \|S(f) - \varphi \circ N(f)\|.$$

In the adaptive case the choice of L_k may depend on $L_1(f), ..., L_{k-1}(f)$, while in the nonadaptive case the L_k are fixed, i.e., independent of $f \in F$. Each method is of the form $S_n^{\text{ad}} = \varphi \circ N$ with an arbitrary $\varphi: \mathbf{R}^n \to Y$. We compare the numbers

$$e^n_{\text{non}}(S|_F) = \inf_{S_n} \Delta_{\max}(S_n)$$

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with the numbers

$$e^n_{\mathrm{ad}}(S\,|_F)=\inf_{S_n^{\mathrm{ad}}} \Delta_{\max}(S_n^{\mathrm{ad}}),$$

where S_n runs through all nonadaptive methods and S_n^{ad} runs through all adaptive methods using an information N consisting of n linear functionals. The adaption problem is the question of whether adaptive methods are better than nonadaptive ones.

We refer the reader to [4] for references about this problem and only mention the well known fact that adaptive methods are not better, up to a factor 2, if the set F is convex and symmetric.

Problems of optimal recovery and n-widths (and also the adaption problem) usually are studied under the assumption that the set F of problem elements is convex and symmetric. In many cases, however, we have a different type of a priori information, such as $f \geqslant 0$ or f is convex. Therefore it is interesting to consider the case where F is only convex and not symmetric, see Novak [4]. For some linear problems $S\colon X \to G$ and convex $F \subset X$ adaptive methods are much better than nonadaptive ones.

In this paper we further compare adaptive and nonadaptive methods and prove a bound on how much adaptive methods can be better. We use new inequalities between Gelfand widths and Bernstein widths, and relations between these widths and optimal error bounds for adaptive and nonadaptive methods, respectively. We also need a result of Sukharev [9] on the optimal recovery of a linear functional on a convex set. Since this paper is a sequel to the recent paper [4], we refer to that paper for further remarks, examples, and references.

2. Diameters of Mappings and Optimal Recovery

We always use the following assumptions and notation. Let X be a normed space over $\mathbf{R}$ and let $F \subset X$ be convex. We assume that $S\colon X \to Y$ is a continuous linear mapping into another normed space Y.

We define several widths of $S\,|_F$. The Kolmogorov width is given by

$$d_n(S\,|_F)=\inf_{Y_n}\ \sup_{f\in F}\ \inf_{g\in Y_n} \|S(f)-g\|, \tag{2.1}$$

where Y_n runs over all n-dimensional affine subspaces of Y. This means, by definition, that Y_n is the translation of a linear subspace U of dimension n, hence Y_n is of the form $Y_n = y + U$ with a $y \in Y$. Every n-dimensional affine subspace Y_n of Y can be written in the form

$$Y_n=\left\{\sum_{i=0}^{n} \alpha_i y_i \;\middle|\; \sum_{i=0}^{n} \alpha_i = 1\right\}$$

with suitable $y_i \in Y_n$. In such a case we write

$$Y_n = \text{aff}\{y_0, y_1, ..., y_n\}$$

and say that Y_n is generated by the y_i. The global Gelfand width is given by

$$d^n_{\text{glob}}(S\,|_F) = \tfrac{1}{2} \cdot \inf_{U_n} \sup_{f \in X} \text{diam}(S(F \cap (U_n + f))), \tag{2.2}$$

while a local variant is given by

$$d^n_{\text{loc}}(S\,|_F) = \tfrac{1}{2} \cdot \sup_{f \in X} \inf_{U_n} \text{diam}(S(F \cap (U_n + f))). \tag{2.3}$$

Here the infimum is taken over closed subspaces U_n of X of codimension at most n. We clearly have

$$d^n_{\text{loc}}(S\,|_F) \leqslant d^n_{\text{glob}}(S\,|_F) \tag{2.4}$$

with equality if F is also symmetric. It is useful in what follows to define Bernstein widths. Here the definition is

$$b_n(S\,|_F) = \sup\{r \mid S(F) \text{ contains an } (n+1)\text{-dimensional ball of radius } r\}. \tag{2.5}$$

Of course these diameters also can be defined for sets by

$$s_n(F) = s_n(S\,|_F),$$

where s_n is one of the widths considered here and $S = Id\colon X \to X$.

There are examples in Novak [4] where the local widths are much smaller than the global widths. In that paper we asked whether there is a bound on how much smaller they can be. In other words, is there an inequality of the form

$$d^n_{\text{glob}}(F) \leqslant c_n \cdot d^n_{\text{loc}}(F) \tag{2.6}$$

with a sequence c_n that is independent of F? If this is the case then of course it would be interesting to know the best constants c_n. Here we prove an inequality of the type (2.6) and apply it to the adaption problem.

First we note a connection between the error bounds e^n_{non} and e^n_{ad} and certain n-widths of $S\,|_F$. Most of these simple relations are already mentioned in Novak [4].

PROPOSITION 1. *Let $F \subset X$ be a convex set and let $S\colon X \to Y$ be linear and continuous. Then*

$$\tfrac{1}{2} \cdot d^n_{\mathrm{glob}}(S\,|_F) \leqslant e^n_{\mathrm{non}}(S\,|_F) \leqslant d^n_{\mathrm{glob}}(S\,|_F), \tag{2.7}$$

and

$$\tfrac{1}{2} \cdot b_n(S\,|_F) \leqslant \tfrac{1}{2} \cdot d^n_{\mathrm{loc}}(S\,|_F) \leqslant e^n_{\mathrm{ad}}(S\,|_F). \tag{2.8}$$

3. AN INEQUALITY BETWEEN DIFFERENT WIDTHS

In this section we prove an inequality between Kolmogorov and Bernstein widths. For the symmetric case, a slightly better estimate can be found in Bauhardt [1], Mityagin and Henkin [3] and in Pinkus [7].

We also should say that better estimates are known in the case where Y is a Hilbert space, see Perelman [5].

LEMMA 1. *Assume that $y_0, ..., y_{n+1} \in S(F)$ and denote by E_k the affine subspace generated by the y_i without y_k,*

$$E_k = \mathrm{aff}\{y_0, ..., y_{k-1}, y_{k+1}, ..., y_{n+1}\}.$$

Let $d(y_k, E_k) = \inf_{z \in E_k} \|y_k - z\|$. Then

$$b_n(S\,|_F) \geqslant \frac{\min_k d(y_k, E_k)}{n+2}.$$

Proof. Consider the midpoint m of the y_i,

$$m = \frac{1}{n+2} \sum_{i=0}^{n+1} y_i,$$

and assume that $\sum_{i \neq k} \alpha_i = 1$. We obtain

$$\begin{aligned}(n+2) \cdot \Big\| m - \sum_{i \neq k} \alpha_i y_i \Big\| &= \Big\| y_k + \sum_{i \neq k} y_i - (n+2) \sum_{i \neq k} \alpha_i y_i \Big\| \\ &= \Big\| y_k - \sum_{i \neq k} \beta_i y_i \Big\|\end{aligned}$$

for some β_i with $\sum_{i \neq k} \beta_i = 1$ and therefore $d(m, E_k) \geqslant 1/(n+2)\, d(y_k, E_k)$. We prove that $S(F)$ contains the $(n+1)$-dimensional ball in

$$Y_{n+1} = \mathrm{aff}\{y_0, y_1, ..., y_{n+1}\}$$

with center m and radius

$$r = \frac{\min_k d(y_k, E_k)}{n+2}.$$

Let $x = m + \sum_{i=0}^{n+1} \alpha_i y_i$ with $\sum_{i=0}^{n+1} \alpha_i = 0$ and $\|\sum \alpha_i y_i\| \leqslant r$. Assume that the smallest α_i is less than $-1/(n+2)$. Then we could conclude that

$$\left\| y_i - \sum_{k \neq i} \beta_k y_k \right\| \leqslant \frac{r}{|\alpha_i|} < r(n+2) \leqslant d(y_i, E_i)$$

for some β_k which sum up to 1. This can not be the case and we obtain $\alpha_i \geqslant -1/(n+2)$. This implies that $x = \sum_{i=0}^{n+1} \lambda_i y_i$, where the $\lambda_i \geqslant 0$ and sum to 1. Thus $x \in S(F)$, since $S(F)$ is convex. ∎

For $n \in \mathbf{N}$ and $y_0, ..., y_{n+1} \in S(F)$ we define

$$V_{n+1}(y_0, y_1, ..., y_{n+1}) = \sup\{\det[l_j(y_i - y_0)]_{i,j=1}^{n+1} \mid l_j \in Y^*, \|l_j\| = 1\}.$$

In the next lemma we collect some properties of these numbers.

LEMMA 2. (1) *$V_{n+1}(y_0, ..., y_{n+1}) > 0$ iff the y_i generate an $(n+1)$-dimensional affine space.*

(2) *V_{n+1} is independent of the ordering of the y_i.*

(3) *$V_{n+1}(y_0 - y, y_1 - y, ..., y_{n+1} - y) = V_{n+1}(y_0, y_1, ..., y_{n+1})$, i.e., this quantity is translation invariant and we may assume that $y_0 = 0$.*

(4)

$$\begin{aligned} V_{n+1}(y_0, ..., y_{n+1})/(n+1) &\leqslant d(y_{n+1}, E_{n+1}) \cdot V_n(y_0, ..., y_n) \\ &\leqslant V_{n+1}(y_0, ..., y_{n+1}). \end{aligned} \tag{3.1}$$

Proof. The proof of the properties (1), (2) and (3) is trivial and therefore it is enough to prove (3.1) for $y_0 = 0$. Assume that $y^* \in E_{n+1}$ is a best approximation of y_{n+1} in E_{n+1}, i.e.,

$$\|y_{n+1} - y^*\| = d(y_{n+1}, E_{n+1}).$$

From the elementary properties of the determinant we obtain that $|\det[l_j(y_i)]_{i,j=1}^{n+1}|$ does not change if y_{n+1} is replaced by $y_{n+1} - y^*$. Because $|l_j(y_{n+1} - y^*)| \leqslant \|y_{n+1} - y^*\|$ we obtain by expansion of the last row

$$|\det[l_j(y_i)]_{i,j=1}^{n+1}| \leqslant (n+1) \|y_{n+1} - y^*\| \mid V_n(y_0, ..., y_n)|$$

which proves the left inequality of (3.1).

To prove the right inequality of (3.1) for the case $y_0 = 0$ assume that $l_1, ..., l_n \in Y^*$ with $\|l_k\| = 1$ for all k. There is an $l_{n+1} \in Y^*$ with $\|l_{n+1}\| = 1$

and $l_{n+1}(y_{n+1}) = d(y_{n+1}, E_{n+1})$ and $l_{n+1}(y_k) = 0$ for $k = 0, ..., n$, since E_{n+1} is a linear subspace because of $y_0 = 0$. Hence we obtain

$$d(y_{n+1}, E_{n+1}) \cdot |\det[l_j(y_i)]_{i,j=1}^{n}| = |\det[l_j(y_i)]_{i,j=1}^{n+1}|$$

and the right inequality of (3.1) follows. ∎

LEMMA 3. *Given* $n \in \mathbf{N}$ and $0 < \varepsilon < 1$, *there are* $y_0, ..., y_{n+1} \in S(F)$ *such that*

$$\min_{k=0,...,n+1} d(y_k, E_k) \geqslant \frac{1}{n+1} d_n(S|_F)(1-\varepsilon)^2. \tag{3.2}$$

Proof. We assume that $S(F)$ contains $(n+2)$ points that generate an $(n+1)$-dimensional affine space since otherwise the statement of the lemma is trivial. We put

$$V_{n+1} = \sup\{ V_{n+1}(y_0, ..., y_{n+1}) \mid y_i \in S(F)\} > 0$$

and choose elements $y_0, ..., y_{n+1} \in S(F)$ such that

$$V_{n+1}(y_0, ..., y_{n+1}) \geqslant (1-\varepsilon)\, V_{n+1}.$$

To prove (3.2) we assume the contrary and, without loss of generality,

$$d(y_{n+1}, E_{n+1}) < \frac{1}{n+1} d_n(S|_F)(1-\varepsilon)^2.$$

By definition of the Kolmogorov widths there is a $y^* \in S(F)$ with $d(y^*, E_{n+1}) \geqslant d_n(S|_F)(1-\varepsilon)$. We obtain the estimate

$$\begin{aligned} V_{n+1}(y_0, ..., y_n, y^*) &\geqslant d(y^*, E_{n+1}) \cdot V_n(y_0, ..., y_n) \\ &\geqslant d_n(S|_F)\, V_n(y_0, ..., y_n)(1-\varepsilon) \\ &> (n+1)(1-\varepsilon)^{-1} d(y_{n+1}, E_{n+1}) \cdot V_n(y_0, ..., y_n) \\ &\geqslant (1-\varepsilon)^{-1}\, V_{n+1}(y_0, ..., y_{n+1}) \geqslant V_{n+1}, \end{aligned}$$

that contradicts $V_{n+1}(y_0, ..., y_n, y^*) \leqslant V_{n+1}$. ∎

The following result follows easily from Lemma 1 and Lemma 3.

THEOREM 1.

$$d_n(S|_F) \leqslant (n+1)(n+2)\, b_n(S|_F). \tag{3.3}$$

4. Bounds for the Adaption Problem

Theorem 1 is about Kolmogorov widths. For applications to the adaption problem we have to replace these diameters by the global Gelfand widths. This can be done using analogous lemmas that reflect the dual situation.

Assume that $l_1, ..., l_{n+1} \in Y^*$ with $\|l_i\| = 1$ for all i and denote by L_k the linear subspace generated by the l_i without l_k,

$$L_k = \text{lin}\{l_1, ..., l_{k-1}, l_{k+1}, ..., l_{n+1}\}.$$

It is useful to define

$$\Delta(l_k, L_k) = \inf_{l \in L_k} \inf_{\alpha \in R} \sup_{y \in S(F)} |l_k(y) - l(y) - \alpha|.$$

The number $\Delta(l_k, L_k)$ indicates how well l_k can be recovered by the other l_i. This follows from a result of Sukharev [9] saying that optimal recovery of a functional l_k using $l_1, ..., l_{k-1}, l_{k+1}, ..., l_{n+1}$ on a convex set F can be achieved by an affine method, hence the optimal error bound is $\Delta(l_k, L_k)$.

We mention, though it is not really important for the following, that the infimum in the definition of $\Delta(l_k, L_k)$ is always attained. The proof consists in three steps and we do not present the details:

Step 1. We can and will assume that $l_1, ..., l_{k-1}, l_{k+1}, ..., l_{n+1}$ are independent on the set $S(F)$.

Step 2. Large values of $|\alpha|$ or of the coefficients $|\alpha_i|$ in $l = \sum_{i \neq k} \alpha_i l_i$ lead to a large error in $\sup_{y \in S(F)} |l_k(y) - l(y) - \alpha|$. Hence it suffices to consider a bounded set of coefficients $(\alpha, \alpha_1, ..., \alpha_{k-1}, \alpha_{k+1}, ..., \alpha_{n+1}) \in \mathbf{R}^{n+1}$.

Step 3. The mapping

$$h\colon (\alpha, \alpha_1, ..., \alpha_{k-1}, \alpha_{k+1}, ..., \alpha_{n+1}) \mapsto \sup_{y \in S(F)} |l_k(y) - l(y) - \alpha| \in \mathbf{R}_0^+ \cup \{+\infty\}$$

could be discontinuous if $S(F)$ is unbounded. Even in this case, however, the set where h is finite is an affine subspace of $\mathbf{R}^{n+1}$ and h is continuous on that set. Hence the existence of an optimal pair (l^*, α^*) follows from compactness.

Lemma 4. *Assume that $l_1, ..., l_{n+1} \in Y^*$ with $\|l_i\| = 1$ for all i. Then*

$$b_n(S\,|_F) \geqslant \frac{\min_k \Delta(l_k, L_k)}{n+1}.$$

Proof. Consider $\tilde{S}\colon X \to \tilde{Y} = \mathbf{R}^{n+1}$ defined by

$$\tilde{S}(f) = (l_1(S(f)), ..., l_{n+1}(S(f))),$$

where $\tilde{Y}$ is equipped with the maximum norm $\|\tilde{y}\|_\infty$. Then we obtain

$$\|\tilde{S}(f_1) - \tilde{S}(f_2)\|_\infty \leqslant \|S(f_1) - S(f_2)\|_Y$$

and hence $b_n(\tilde{S}\,|_F) \leqslant b_n(S\,|_F)$. Let $k \in \{1, ..., n+1\}$ and $c = \min_i \Delta(l_i, L_i)$ and assume that $0 < \varepsilon < c$. Then there is a $\tilde{y}_k \in \tilde{S}(F)$ such that the whole interval

$$I_k := \{\tilde{y}_k + \lambda e_k \mid |\lambda| \leqslant c - \varepsilon\}$$

is contained in $\tilde{S}(F)$. By $e_k \in \mathbf{R}^{n+1}$ we mean the vector with coordinates δ_{ik}, $i = 1, ..., n+1$. This follows from the result of Sukharev [9] mentioned above. We claim that $\tilde{S}(F)$ contains the $(n+1)$-dimensional ball with radius

$$r = \frac{\min_k \Delta(l_k, L_k) - \varepsilon}{n+1}$$

and midpoint

$$m = \frac{1}{n+1} \sum_{k=1}^{n+1} \tilde{y}_k.$$

Let

$$y = \frac{1}{n+1} \sum_{k=1}^{n+1} \tilde{y}_k + \sum_{k=1}^{n+1} \delta_k e_k$$

with $|\delta_k| \leqslant r$. Then $\tilde{y}_k + (n+1)\,\delta_k e_k \in I_k \subset \tilde{S}(F)$ and y is a convex combination of these elements of $\tilde{S}(F)$. We conclude that $b_n(S\,|_F) \geqslant b_n(\tilde{S}\,|_F) \geqslant \min_k \Delta(l_k, L_k)/(n+1)$. ∎

For $n \in \mathbf{N}$ and $l_1, ..., l_{n+1} \in Y^*$ we define

$$W_{n+1}(l_1, ..., l_{n+1}) = \sup\{\det[l_j(y_i - y_0)]_{i,j=1}^{n+1} \mid y_0, ..., y_{n+1} \in S(F)\}.$$

We collect some properties of these numbers.

LEMMA 5. (1) *$W_{n+1}(l_1, ..., l_{n+1}) > 0$ for some set of l_i iff $S(F)$ contains $y_0, ..., y_{n+1}$ that generate a $(n+1)$-dimensional affine space.*

(2) *W_{n+1} is independent of the order of the l_i.*

(3)

$$\begin{aligned} W_{n+1}(l_1, ..., l_{n+1}) &\leqslant (2n+2) \cdot \Delta(l_{n+1}, L_{n+1}) \cdot W_n(l_1, ..., l_n) \\ &\leqslant (2n+2) \cdot W_{n+1}(l_1, ..., l_{n+1}). \end{aligned} \tag{4.1}$$

Proof. The proof of (1) and (2) is trivial. To prove (4.1) assume that the pair $l^* \in L_{n+1}$ and $\alpha^* \in \mathbf{R}$ yield a best approximation of l_{n+1} in L_{n+1}, i.e.,

$$\Delta(l_{n+1}, L_{n+1}) = \sup_{y \in S(F)} |l_{n+1}(y) - l^*(y) - \alpha^*|.$$

The existence of an optimal pair (l^*, α^*) is not really needed in the following because we also could work with a pair (l^*, α^*) which is "almost" optimal.

We prove the left inequality of (4.1). First we can replace l_{n+1} in $|\det[l_j(y_i - y_0)]_{i,j=1}^{n+1}|$ by $l_{n+1} - l^*$ without changing the determinant. Now we add and subtract α^* to the elements of the last column. We obtain, expanding by the last column of the matrix, the desired result. The factor two comes in because of the $y_i - y_0$ leading to two sums.

To prove the right inequality we first remark that $W_n(l_1, ..., l_n)$ does not change if a l_k is replaced by $l_k + l$ with $l \in \operatorname{lin}\{l_1, ..., l_{k-1}\}$. Hence we find for a given ε with $0 < \varepsilon < 1$ elements $y_0, ..., y_n$ such that

$$|\det[l_j(y_i - y_0)]_{i,j=1}^{n}| > W_n(l_1, ..., l_n) \cdot (1 - \varepsilon)$$

and we may assume that $l_j(y_i - y_0) = 0$ for $i < j$. In particular, we replace l_{n+1} by $\tilde{l}_{n+1} = l_{n+1} - l$ with $l \in L_{n+1}$ such that $\tilde{l}_{n+1}(y_i - y_0) = 0$ for $i = 1, ..., n$. So we obtain the desired estimate by

$$\begin{aligned} &W_{n+1}(l_1, ..., l_{n+1}) \\ &\quad \geqslant W_n(l_1, ..., l_n) \cdot (1 - \varepsilon) \inf_{l \in L_{n+1}} \sup_{y_{n+1} \in S(F)} |(l_{n+1} - l)(y_{n+1} - y_0)| \end{aligned}$$

because of

$$\inf_{l \in L_{n+1}} \sup_{y_{n+1} \in S(F)} |(l_{n+1} - l)(y_{n+1} - y_0)| \geqslant \Delta(l_{n+1}, L_{n+1}). \quad \blacksquare$$

The analog to Lemma 3 is the following.

LEMMA 6. *Given* $n \in \mathbf{N}$ *and* $0 < \varepsilon < 1$, *there are* $l_1, ..., l_{n+1} \in Y^*$ *with* $\|l_k\| = 1$ *for all* k *such that*

$$\min_{k=1, ..., n+1} \Delta(l_k, L_k) \geqslant \frac{1}{2n+2} d^n_{\text{glob}}(S|_F)(1 - \varepsilon)^2. \tag{4.2}$$

Proof. We assume that there are $l_1, ..., l_{n+1} \in Y^*$ such that $W_{n+1}(l_1, ..., l_{n+1}) > 0$ since otherwise the statement of the lemma is trivial. We put

$$W_{n+1} = \sup\{W_{n+1}(l_1, ..., l_{n+1}) \mid \|l_k\| = 1, k = 1, ..., n+1\}$$

and choose $l_1, \ldots, l_{n+1} \in Y^*$ with $\|l_k\| = 1$ for all k such that

$$W_{n+1}(l_1, \ldots, l_{n+1}) \geqslant (1-\varepsilon) \cdot W_{n+1}.$$

To prove (4.2) it is enough to show $\Delta(l_{n+1}, L_{n+1}) \geqslant (2n+2)^{-1} d^n_{\text{glob}}(S\,|_F)(1-\varepsilon)^2$. We assume the contrary, i.e.,

$$\Delta(l_{n+1}, L_{n+1}) < (2n+2)^{-1} d^n_{\text{glob}}(S\,|_F)(1-\varepsilon)^2. \tag{4.3}$$

We define

$$U_n = \{ f \in X \mid l_i(S(f)) = 0,\ i = 1, \ldots, n \}.$$

Then $U_n \subset X$ is a closed subspace with codimension at most n and therefore, by definition of the Gelfand numbers,

$$d^n_{\text{glob}}(S\,|_F) \leqslant \tfrac{1}{2} \sup_{f \in X} \operatorname{diam}(S(F \cap (U_n + f))).$$

The right side of the last display can also be written as

$$\tfrac{1}{2} \sup_{y \in R^n} \operatorname{diam}\{ S(g) \mid g \in F,\ l_i(S(g)) = y_i,\ i = 1, \ldots, n \}.$$

Hence we find a $\tilde{y} \in \mathbf{R}^n$ and $g_1, g_2 \in F$ such that $l_i(S(g_1)) = l_i(S(g_2))$ for all i and

$$(1-\varepsilon) \cdot d^n_{\text{glob}}(S\,|_F) \leqslant \tfrac{1}{2} \|S(g_1) - S(g_2)\|.$$

By Hahn-Banach there is a $l^* \in Y^*$ with $\|l^*\| = 1$ such that $|l^*(S(g_1)) - l^*(S(g_2))| = \|S(g_1) - S(g_2)\|$. Because of $l_i(S(g_1)) = l_i(S(g_2))$ for $i = 1, \ldots, n$ we obtain

$$\Delta(l^*, L_{n+1}) \geqslant \tfrac{1}{2} |l^*(S(g_1)) - l^*(S(g_2))|.$$

Putting these formulas together we obtain

$$\Delta(l^*, L_{n+1}) \geqslant d^n_{\text{glob}}(S\,|_F) \cdot (1-\varepsilon).$$

Using the assumption (4.3) and (4.1) we conclude that

$$\begin{aligned} W_{n+1}(l_1, \ldots, l_n, l^*) &\geqslant \Delta(l^*, L_{n+1}) \cdot W_n(l_1, \ldots, l_n) \\ &\geqslant d^n_{\text{glob}}(S\,|_F)\, W_n(l_1, \ldots, l_n)(1-\varepsilon) \\ &> (2n+2)(1-\varepsilon)^{-1}\, \Delta(l_{n+1}, l_{n+1})\, W(l_1, \ldots, l_n) \\ &\geqslant (1-\varepsilon)^{-1}\, W(l_1, \ldots, l_{n+1}) \geqslant W_{n+1} \end{aligned}$$

that contradicts $W_{n+1}(l_1, \ldots, l_n, l^*) \leqslant W_{n+1}$, i.e., the formula (4.3) must be wrong. ∎

THEOREM 2.

$$d^n_{\mathrm{glob}}(S\mid_F)\leqslant 2(n+1)^2\, b_n(S\mid_F).$$

This result follows easily from Lemma 4 and Lemma 6. By Proposition 1 and Theorem 2 we immediately obtain a bound on how much adaptive methods can be better than nonadaptive ones.

THEOREM 3.

$$e^n_{\mathrm{non}}(S\mid_F)\leqslant 4(n+1)^2\, e^n_{\mathrm{ad}}(S\mid_F).$$

Remarks. (a) We conjecture that the bounds are not optimal and that $e^n_{\mathrm{non}}(S\mid_F)=O(n\cdot e^n_{\mathrm{ad}}(S\mid_F))$ is true in general. Related conjectures (about Bernstein diameters and Kolmogorov diameters in the symmetric case) are due to Mityagin and Henkin [3] and also can be found in [1], [6] and [7]. These conjectures are still unproved.

(b) Examples where adaptive methods are much better than nonadaptive ones are described in Novak [4]. Similar examples (with function values instead of arbitrary functionals) can also be found in Korneichuk [2] and Sonnevend [8].

5. Open Problems

We mention three open problems connected with the results of this paper:

1. Improve the bounds of Theorem 1, Theorem 2, and Theorem 3.
2. Study the adaption problem in the case where only a restricted set of linear functionals $L_i\in X^*$ (such as function evaluations) is allowed.
3. Study the adaption problem for special operators $S\colon F\to Y$, where, for example, S is the solution operator of an integral equation of the first or second kind.

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